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A PROBLEM IN MECHANICAL FLIGHT.

By Prof. George E. Curtis, Washington, D. C.

In 1883 Lord Rayleigh pointed out* that whenever a bird pursues its course for some time without working its wings, we must conclude "either (1) that the course is not horizontal; (2) that the wind is not horizontal; or (3) that the wind is not uniform. It is probable that the truth is usually represented by (1) or (2);" and adds "but the question I wish to raise is whether the cause suggested by (3) may not sometimes come into operation."

It will be noticed that case (1) of this analysis is not adequately defined, since it is not expressed like the others in terms of the wind. A further examination of the article discloses that this case is that in which the bird by ascending and descending takes advantage of *winds of different velocity at different altitudes*. Lord Rayleigh expressed his distrust of the adequacy of either (1) or (2) to account for the observed facts, but did not investigate the efficiency of (3). The matter has rested at this point until a very recent period. In a memoir recently published by Professor S. P. Langley,† it is shown that the condition represented by (3) always exists; that the wind is never a homogeneous current but consists of a continued series of rapid pulsations varying indefinitely in amplitude and period. The periods are measured by seconds rather than by minutes, and the amplitudes increase with the wind velocity. Having established the fact that the wind is not uniform, the author then gives a popular explanation of the mechanical principles which enable the bird to utilize such pulsations in maintaining its flight without working its wings, or expending energy. This may be accomplished by a succession of ascents and descents; the ascents being made during the wind gusts, and the descents during the lulls.

Manifestly, however, in order to show that this third case of Lord Rayleigh's analysis is the one *actually employed* by the bird in soaring, it is necessary to show that the pulsations in addition to being qualitively applicable are also quantitively sufficient. The bird must rise as high in his ascents as he falls in his descents, and this equality of altitude must be shown to be possible for any assigned wind pulsations in which soaring occurs.

With this application in view, the present paper seeks to determine the course in the air of a free heavy plane subjected to definite wind pulsations.

* *Nature*, April 5, 1883.

† The internal work of the wind. By S. P. Langley. Smithsonian Contributions to Knowledge. Washington, 1893.

Since the actual pulsations of the wind exhibited by the anemometer records are in every case too complex to be treated analytically at this stage of the investigation, I shall follow a suggestion made for this purpose by Professor Langley, and consider that a homogeneous wind blows for a very short period at a uniform rate, then that there is an equal period of calm, and so on alternately. The body immersed in the wind is supposed to be a material plane whose front edge (transverse to the wind) is six times its width, and whose surface is n square feet to the pound (of its weight). For numerical computation the periods of wind and of calm will first be assumed to be five seconds each. Let us take up the investigation of the motion at the beginning of a period of calm, when we will suppose the plane to be momentarily at rest at a given height in the air, and to be capable of changing its angle without expenditure of energy. Since the initial angle which the plane assumes is not conditioned, we will suppose it to be 60° below the horizontal, which is approximately that sometimes taken by birds in the beginning of a rapid descent, and that under the force of gravity the plane glides down the air in the curve of quickest descent (the inverted cycloid) until its course becomes horizontal at the vertex.

If s = any distance measured along the curve, and
a = radius of the generating circle,
the equation of the cycloid, in which the lowest point of the curve is taken as the origin and the axis of y is vertical, is

$$s^2 = 8\,ay\,, \text{ and } \frac{ds}{dy} = \sqrt{\frac{2a}{y}}\,.$$

Since the initial angle of the plane with the horizon is 60°, we have

$$\frac{dy_1}{ds} = \sin 60^\circ = \tfrac{1}{2}\sqrt{3}\,,$$

y_1 being the ordinate of the initial position ;

$$\therefore \quad \frac{ds}{dy_1} = \frac{2}{\sqrt{3}} = \sqrt{\frac{2a}{y_1}}\,, \quad \text{or} \quad \frac{2a}{y_1} = \frac{4}{3}\,,$$

an equation between a, the radius of the generating circle, and the distance of the fall y_1. Let the fall of the plane be 36 feet; we then have a cycloid in which $a = 24$ feet.

The time of fall from any point on an inverted cycloid to the vertex is

$$t = \pi\sqrt{\frac{a}{g}}\,.*$$

* Tait and Steele: Dynamics of a particle, p. 173.

Whence, in this case, if the air reacted like a smooth solid on the gliding plane, and the plane moved tangent to itself, we should have

$$t = \pi \sqrt{\tfrac{24}{32}} = 2.72 \text{ secs.},$$

and the velocity of the plane, which is given by the expression

$$\left[\frac{ds}{dt}\right]^2 = \frac{g}{4a}(s_1^2 - s^2)$$

would become at the vertex of the cycloid

$$\left[\frac{ds}{dt}\right]^2 = \frac{g}{4a}s_1^2, \quad \text{or} \quad V = \sqrt{2gy_1} = 48 \text{ feet per second.}$$

But since the air does not react like a solid, the plane in order to take the assumed cycloidal path cannot move forward in its own plane, but must make a small angle a with the path, such that the sustaining component of the air pressure shall be just sufficient to balance the force of gravity to the required extent. This resistance of the air will cause a lengthening of the time and a diminution of the velocity with which the plane reaches the vertex of the cycloid.

The equation of motion is

$$\frac{d^2s}{dt^2} = -g \cdot \frac{s}{4a} + kng\left[\frac{ds}{dt}\right]^2 F(a) \sin a,$$

in which s is measured from the vertex. The first term of the second member is the acceleration due to gravity, and the second term is the retardation due to the resistance of the air acting on the plane at the angle a. k = the constant of air pressure = 0.00166 lbs. per sq. foot for a velocity of 1 foot per second.

Let $n = 2.3$, which is a ratio of surface (in sq. feet) to weight (in lbs.) found in certain species of soaring birds. $F(a)$ = the ratio of pressure on an inclined plane to the pressure on a normal plane, determined experimentally for rectangles of various shapes in *Experiments in Aerodynamics* by S. P. Langley.

The angle a probably varies within small limits at different parts of the cycloidal path, but for a first approximation in order to be able to integrate, it will here be taken as constant.

Let

$$kng\, F(a) \sin a = c.$$

Then

$$\frac{d^2s}{dt^2} - c\left[\frac{ds}{dt}\right]^2 = -g\frac{s}{4a};$$

whence, multiplying through by $e^{-2cs}ds$, we obtain

$$2\frac{d^2s}{dt}\left[\frac{ds}{dt}\right]e^{-2cs} - 2\,ce^{-2cs}\left[\frac{ds}{dt}\right]^2 ds = -\frac{g}{2a}\,se^{-2cs}ds\,.$$

The first member is now a complete differential of which the integral is $\left[\frac{ds}{dt}\right]^2 e^{-2cs}$;

$$\therefore\quad \left[\frac{ds}{dt}\right]^2 e^{-2cs} = -\frac{g}{2a}\int se^{-2cs}ds + \text{const.}$$

Integrating the second member by parts,

$$\left[\frac{ds}{dt}\right]^2 = \frac{g}{2a}\left[\frac{1+2cs}{4c^2}\right] + e^{2cs}\,.\,\text{const.}$$

To determine the constant, we have $s = s_1$ when $\frac{ds}{dt} = 0$; hence

$$\text{const.} = -\frac{g}{2a}\cdot\frac{1+2cs_1}{4c^2}\,.\,e^{-2cs_1}\,,$$

and

$$\left[\frac{ds}{dt}\right]^2 = \frac{g}{2a}\left[\frac{1+2cs}{4c^2} - \left[\frac{1+2cs_1}{4c^2}\right]e^{2c(s-s_1)}\right].$$

By putting $s = 0$, the velocity at the vertex of the cycloid is given by the equation

$$\left[\frac{ds}{dt}\right]^2_{s=0} = \frac{g}{2a}\left[\frac{1}{4c^2} - \left[\frac{1+2cs_1}{4c^2}\right]e^{-2cs_1}\right]$$

In order to evaluate this velocity we must determine the numerical values of the constants.

From the equation of the cycloid,

$$s_1 = \sqrt{8ay_1} = 48\sqrt{3} = 83.1\,.$$

c was substituted for the expression $kng\,F(a)\sin a$ in which we must assign a value for the angle a.

From the numerical results obtained in later portions of this analysis, 2° is here adopted as an approximate mean value for a, and from *Experiments in Aerodynamics*, p. 62, $F(a) = F(2°) = 0.15$. With these values and others already given,

$$c = 0.00166\,(2.3)\,(32\tfrac{1}{6})\,(0.15)\,(0.0349) = 0.000643\,,$$

$$2cs_1 = 0.10687\,,\quad e^{2cs_1} = 1.113\,,\quad \frac{ds}{dt} = 47.2 \text{ feet per sec.}$$

Comparing this result with that above given for a body sliding down a solid frictionless cycloid, we find that the velocity at the vertex is 0.8 foot per second smaller, or 47.2 instead of 48.0 feet per second.

The time of reaching the vertex is given by the integral

$$t = \int_{s=s_1}^{s=0} \frac{ds}{\sqrt{\frac{g}{2a}\left[\frac{1+2cs}{4c^2} - \left(\frac{1+2cs_1}{4c^2}\right)e^{2c(s-s_1)}\right]}} .$$

This expression is not directly integrable, but its value can be computed approximately by the method of quadratures. A computation by this method of the time between $s = 0$ and $s_1 = 65$ (the portion of the path for which the retardation is appreciable), gives .03 second more than the time given by the simpler formula; making the time for describing the whole path 2.75 seconds instead of 2.72 seconds.

There still remain then 2.25 seconds of the assigned period of calm. We may assume that the plane, after reaching the vertex of the cycloid with a horizontal velocity v, at once assumes an upward inclination of 7°, the front edge being elevated.

The vertical component of air pressure on the plane due to its own motion will be*

$$kA\,V^2F(a)\cos 7^\circ ,$$

in which k = the constant of air pressure = .00166 lbs. per sq. foot;

$A = nW$ = area of surface in sq. feet;

V = relative velocity of air and plane in feet per second;

a = angle of inclination between plane and path of advance;

$F(a)$ = ratio of pressure on an inclined plane to the pressure on a normal plane, determined experimentally for rectangles of various shapes in *Experiments in Aerodynamics*.

At the vertex of the cycloid the path of the plane is horizontal, and hence initially

$a = 7°$,

$V = 47.2$ feet per second, as just determined, and

$A = 2.3\,W$ (weight of plane), by hypothesis.

For $a = 7°$, $F(a)$, for a plane whose length is 6 times its width, equals 0.36. (*Experiments in Aerodynamics*, Diagram, p. 62.)

Substituting these values in the expression just given, we find that the vertical upward component of pressure on the plane is 3.15 W, or 3.15 times

* See *Experiments in Aerodynamics*, p. 60.

the weight of the plane, and therefore materially exceeds the force of gravity. The motion of the plane will consequently have a vertical upward component.

As the plane begins to rise, the angle between the plane and the wind of advance diminishes, and hence the vertical component of pressure diminishes. Uniform motion will be attained when the vertical component of pressure equals the weight.

For this condition, we shall have the equation

$$W = 0.00166 \times 2.3 \times W \times 47.22 \times F(a) \cos 7°$$

to determine a.

$$1 = 3.70 \times 2.3 \times F(a) \cos 7° ,$$

$$F(a) = .118 ,$$

$$a = 1\tfrac{1}{2}°. \quad (\textit{Experiments in Aerodynamics}, \text{p. } 62)$$

This result means that the plane will take up a path making an angle of $5\frac{1}{2}$ degrees with the horizontal, while its own angle is 7°. This cannot, in fact, be done instantaneously, but the time required is so short that for our purpose we may consider this to be the condition at the outset of the motion. The initial velocity of 47.2 feet per second will now be subject to diminution from the resistance of the air due to the angle of $1\frac{1}{2}°$ between the plane and the wind of advance. As the velocity diminishes, the angle a must increase in order that the equation between weight and vertical pressure shall be preserved. The increase of a in turn brings about an increase in the horizontal resistance, and hence it follows that the velocity of the plane decreases at an increasing rate.

The differential equations of motion are

$$\frac{W}{g} \cdot \frac{d^2x}{dt^2} = - kn\, W V^2 F(a) \sin 7° , \tag{1}$$

$$\frac{W}{g} \cdot \frac{d^2y}{dt^2} = + kn\, W V^2 F(a) \cos 7° - W, \tag{2}$$

in which V and a are variables.

In order to integrate, let us divide the period into a series of such short intervals that for each of these a may be assumed without appreciable error to be constant.

For the first interval, let $a = 1° 30'$. Then the angle of path with the horizon will be $5° 30'$, and we shall have

$$V = \frac{dx}{dt} \sec (7° - a) = \frac{dx}{dt} \sec 5° 30',$$

and

$$\frac{d^2x}{dt^2} = -kngF(a)\sin 7^\circ \left[\frac{dx}{dt}\right]^2 \sec^2 5^\circ\ 30'.$$

To integrate this, put

$$c' = kng\,F(a)\sin 7^\circ \sec^2 5^\circ\ 30', \quad \text{and} \quad w = \frac{dx}{dt};$$

then

$$\frac{dw}{dt} = -c'w^2, \text{ or } \frac{1}{w^2}\frac{dw}{dt} = -c';$$

whence, integrating,

$$\frac{1}{w} = c't + \text{const.}$$

Substituting the value of w,

$$\frac{1}{\frac{dx}{dt}} = c't + \text{const.}$$

When $t = 0$, $\frac{dx}{dt} = V_0 \cos(7^\circ - a)$; whence, const. $= \frac{1}{V_0\cos(7^\circ - a)}$, and we have

$$\frac{dx}{dt} = \frac{V_0\cos(7^\circ - a)}{1 + c'tV_0\cos(7^\circ - a)}. \tag{3}$$

Integrating again, we have

$$x = \frac{1}{c'}\log\left[c't + \frac{1}{V_0\cos(7^\circ - a)}\right] + \text{const.}$$

When $t = 0$, $x = 0$; whence const. $= -\frac{1}{c'}\log\frac{1}{V_0\cos(7^\circ - a)}$.

$$\therefore \quad x = \frac{1}{c'}\log\left[\frac{c't + \frac{1}{V_0\cos(7^\circ - a)}}{\frac{1}{V_0\cos(7^\circ - a)}}\right] = \frac{1}{c'}\log\left[1 + c'tV_0\cos(7^\circ - a)\right]. \tag{4}$$

Substituting the assigned numerical values for the quantities in the equation for c', we have

$$c' = .00166 \times 2.3 \times 32.18 \times F(1\tfrac{1}{2}^\circ)\sin 7^\circ . \sec^2 5\tfrac{1}{2}^\circ .$$

As we have already seen, $F(1\frac{1}{2}^\circ) = 0.118$; hence,

$$c' = 0.00177;$$

$$\frac{dx}{dt} = \frac{47.2\cos 5\frac{1}{2}^\circ}{1 + .0835\,t\cos 5\frac{1}{2}^\circ}, \quad V = \frac{47.2}{1 + .0835\,t\cos 5\frac{1}{2}^\circ}.$$

The period for which the integration is to be made is 2.25 seconds, and this interval is so short that we may evaluate the equations for the whole of it without making a material error. Placing therefore $t = 2.25$, we have

$$V = 40 \text{ feet per second},$$

$$x = 97 \text{ feet}.$$

The altitude gained will be 97 tan $5\frac{1}{2}° = 9.3$ feet.

At the beginning of the second interval of 5 seconds, the wind, by hypothesis, begins to blow with a uniform velocity. For the purposes of the present example, let this velocity be 36 feet per second. The relative velocity of wind and plane will now be the geometrical resultant of their respective velocities. Let the plane maintain its constant angle of 7° with the horizon. The vertical upward component of air pressure will again give a vertical component to the plane's motion, and the plane will begin to ascend until the angle of path is such that the vertical upward component of pressure is equal to the weight. For a condition of uniform motion we shall have, as before, the equation

$$W = kn\, W V^2 F(a) \cos 7° ,$$

in which, now,

V is the relative velocity of wind and plane ;

a is the angle between the plane and the direction of V.

It will be seen that the preceding equation for evaluating $F(a)$ contains the relative velocity V, which is itself a function of a. But since a change in a makes only a very small change in V, we may compute V by assuming an approximate value of a, and then compute $F(a)$ from the equation, in order to obtain a more accurate value.

Assuming $a = 1°$, we obtain by a solution of the triangle ABC, in which AB represents the horizontal velocity of the wind, CB the velocity of the plane, and AC the relative velocity V,

$$V = 75.6 \text{ feet per second}.$$

Let this initial value of V be designated V_0. We have then,

$$F(a) = \frac{1}{kn\, V_0^2 \cos 7°} = 0.046 ;$$

$$a = \tfrac{1}{2}° . \quad \text{(Experiments in Aerodynamics, p. 62.)}$$

If we substitute this value in the differential equation (1) already given for $\frac{d^2x}{dt^2}$, and notice that now $\frac{dx}{dt}$ denotes *relative* and not *absolute* velocity,

we have

$$\frac{d^2x}{dt^2} = -g\frac{V^2}{V_0^2}\tan 7^\circ = -g\frac{\tan 7^\circ}{V_0^2}\cdot\left[\frac{dx}{dt}\right]^2\sec^2(7^\circ - a)$$

Integrating as before, putting

$$g\frac{\tan 7^\circ}{V_0^2}\sec^2(7^\circ - a) = c' = 0.000695\,,$$

we have

$$\frac{dx}{dt} = \frac{V_0\cos(7^\circ - a)}{1 + c't\,V_0\cos(7^\circ - a)}\,,$$

and

$$x = \frac{1}{c'}\log\left[1 + c't\,V_0\cos(7^\circ - a)\right].$$

Since the integration has been made on the condition that t be taken as a short interval for which a is sensibly constant, we will divide the whole period of 5 seconds into two equal parts, computing the position of the plane at the end of $2\frac{1}{2}$ seconds, and with its velocity at this position, make a second computation for the remaining $2\frac{1}{2}$ seconds.

With $t = 2\frac{1}{2}$, $\dfrac{dx}{dt} = 66.9$;

$$x = 177 \text{ feet}.$$

The relative velocity at the end of the interval is

$$V = \frac{dx}{dt}\sec 6\tfrac{1}{2}^\circ = 67.3\,.$$

The *absolute* horizontal distance traversed by the plane $= 177 - 2\frac{1}{2}(36) = 87$ feet.

The angle of the path of plane with the horizon becomes known after computing the value of the angle C in the triangle ABC. The known parts are now $A = 6^\circ\ 30'$, $AB = 36$, and AC, which is given a mean value between the value $V_0 = 75.6$ at the beginning of the period, and $V = 67.3$ at the end. a being taken as $0^\circ\ 30'$, the angle of the path with the horizon is found to be 13°. The ascent made by the plane in two and a half seconds will be

$$y = 87\tan 13^\circ = 20.1\,.$$

Taking now the value $V = 67.3$ as the initial relative velocity V_0, for the remaining two and a half seconds, and repeating the computation we have the following result:

$$F(a) = 0.058\,; \qquad a = 0^\circ\ 40'\,; \qquad c' = 0.00088$$

$$\frac{dx}{dt} = 58.5\,; \qquad x = 157 \text{ feet}\,; \qquad V = 58.9\,.$$

The absolute horizontal distance travelled $= 157 - 2.5 \times (36) = 67$ feet. The mean angle of path computed as before, is found to be $14° 36'$, and the ascent made by the plane is $67 \tan 14° 36' = 17.5$ feet.

The total height gained by the plane after reaching its lowest point at the vertex of the cycloid, is the sum of the separate heights gained in the three intervals for which we have made the computation, namely, 9.3, 20.1 and 17.5 feet, making 46.9 feet in all. (This result is slightly greater than would be given by a perfect integration, since a is not absolutely constant for the intervals used.) Thus, without any internal source of energy, during the 10 seconds of alternate calm and wind, the plane has in the first 2.75 seconds made a descent of 36 feet, and in the remaining 7.25 seconds has risen 46.9 feet, travelling at the same time horizontally a distance of 251 feet, of which 154 feet is made against the wind. In addition, the relative velocity of the plane and the wind (58.9 feet per second) at the end of this period is sufficient, if the wind continue with the same velocity, to yield a considerable further ascent before the vertical component of pressure is reduced to such an extent that it no longer exceeds the weight of the plane under the constant angle of inclination adopted.

Evidently, the trajectory described by the plane during the first period of ten seconds, can be repeated indefinitely so long as the wind possesses the assigned pulsations. The problem stated at the outset of the paper has therefore been solved for a single case in which the wind pulsations are of a simple character. From the results obtained in the course of the solution, a number of interesting generalizations are immediately deducible as to the effect of a change in the value of the constants which enter the formulæ. In every case a change in the numerical values affects the soaring plane favorably or unfavorably, and for each one limiting values can be found beyond which soaring will be impossible, all the other conditions remaining unchanged.

Up to a certain limit soaring is facilitated by a decrease in the value of n, the number of square feet of sustaining surface to a pound of weight in the soaring plane or bird. For a given area of wing, the heavier bird possesses the greater momentum, and sails with greater steadiness amid all the varying gusts of the pulsating wind. With respect to the wind, soaring is facilitated by an increase in the amplitude, a, of the wind pulsations and by a decrease in the interval, t, of their recurrence. It may be observed also that in general an advantage accrues when the period of lull is shorter than the period of gust. The relative numerical values of n, a, and t are the principal factors affecting the possibility of soaring in any specific case. It should also be remarked here that certain forms of curved surface, like the bird's wing, are so much more advantageous than planes, that soaring is possible for

them in wind pulsations which would not sustain the simple plane ; but the pressure constants applicable to such surfaces have not yet been determined.

In the example here solved, the wind-lulls have been assumed as actual calms, and the plane has moved in a straight forward path. This condition seldom occurs in nature, for even in intervals of relative calm there is generally some wind, and for a straightforward path this wind tends to strike the descending body on the upper side of its supporting surface. I have made the suggestion that the soaring bird avoids such a position when possible, and that this is a reason for the circling movement so generally adopted in soaring flight.

NOTE ON THE EXPANSION OF A FUNCTION.

By Prof. W. H. Echols, Charlottesville, Va.

1. The following simple method of obtaining the expansion of a function in integral powers of the variable I have not seen in print.

Differentiate the function

$$\frac{fx - fa}{x - a} \tag{1}$$

n times with respect to a, by applying Leibnitz's formula for forming the nth derivative of a product, and we obtain at once

$$fx - fa - (x-a)f'a - \ldots - \frac{(x-a)^n}{n!}f^n a$$

$$= \frac{(x-a)^{n+1}}{n!}\left[\frac{d}{da}\right]^n\left[\frac{fx - fa}{x-a}\right] \tag{2}$$

in which identity, the member on the right represents the remainder.

Since the above is a mere algebraical identity, it is true, whether the argument be real or complex. The only consideration we have to make in regard to the function f, is that its n derivatives are finite and existent at a.

2. To throw the remainder into familiar form we make use of the theorem of mean value, which may be presented thus :—

We have,

$$\frac{fx - fa}{x - a} = \frac{1}{m}\left[\frac{fx - fx_1}{\Delta x} + \frac{fx_1 - fx_2}{\Delta x} + \ldots + \frac{fx_{n-1} - fa}{\Delta x}\right], \tag{3}$$

in which we have $x - a = m\Delta x$, and have assumed that along the line between x and a the function f is finite at the points $x_r = x + r\Delta x$ ($r = 0, \ldots, n$). Evidently the second member of (3) is the mean of the m ratios in the parenthesis ; if the arguments are real, the left member lies between the greatest and least of these ratios. If the derivative of the function f is not infinite throughout the interval between x and a we may make m as great as we choose, and in the limit, when $n = \infty$, we find that the ratio $(fx - fa)/(x - a)$ must lie between the greatest and least values of the derivative of f, in the interval (x, a). If the derivative be continuous between x and a, or only continuous between those two values of the argument in the interval (x, a) at which it takes the greatest and least values, then evidently there must be a value of the

argument u, between these values, such that

$$\frac{fx - fa}{x - a} = f'u \,. \tag{4}$$

If the argument of the function be complex, then we have from (3)

$$\left|\frac{fx - fa}{x - a}\right| = \frac{1}{m}\left|\Sigma \frac{\Delta fx}{\Delta x}\right|$$

$$< \frac{1}{m} \Sigma \left|\frac{\Delta fx}{\Delta x}\right| = \frac{\rho}{m} \Sigma \left|\frac{\Delta fx}{\Delta x}\right| = \rho\, |f'u| \,.$$

In which ρ is a real positive number less than unity, and u is some point on the straight line between x and a. Let φ be the amplitude of the member on the left and ψ that of $f'u$, then

$$\frac{fx - fa}{x - a} = \rho\, e^{i(\phi - \psi)} f'u = \lambda f'u \tag{5}$$

wherein mod λ is less than unity. This is then the theorem of mean value, and when the argument is real $\lambda \quad 1$.

The remainder (2) may now be written

$$\frac{(x-a)^{n+1}}{n!}\left[\frac{d}{da}\right]^n \left[\frac{fx - fa}{x - a}\right] = \frac{(x-a)^{n+1}}{n!}\left[\frac{d}{da}\right]^n (\lambda f'u)$$

$$= \frac{(x-a)^{n+1}}{n!}\left[\frac{d}{da}\right]^n \lambda f'\,[x + \theta(a - x)], \tag{6}$$

θ being a real positive number less than unity.

3. We now show that the differentiation indicated in (6) may be performed just as though λ and θ were absolute constants with respect to a. As in Todhunter's Calculus and elsewhere, let the right member of (2) be represented by $R(a)$. We have $R(x) = 0$, and by differentiating with respect to a, we get

$$R'(a) = -\frac{(x-a)^n}{n!} f^{n+1}a \,.$$

Applying the theorem of mean value to the function $R(a)$, we have

$$R(a) = \lambda' (a - x)\, R'(u')$$

$$= \frac{(x-a)(u'-a)^n}{n!} \lambda' f^{n+1}u$$

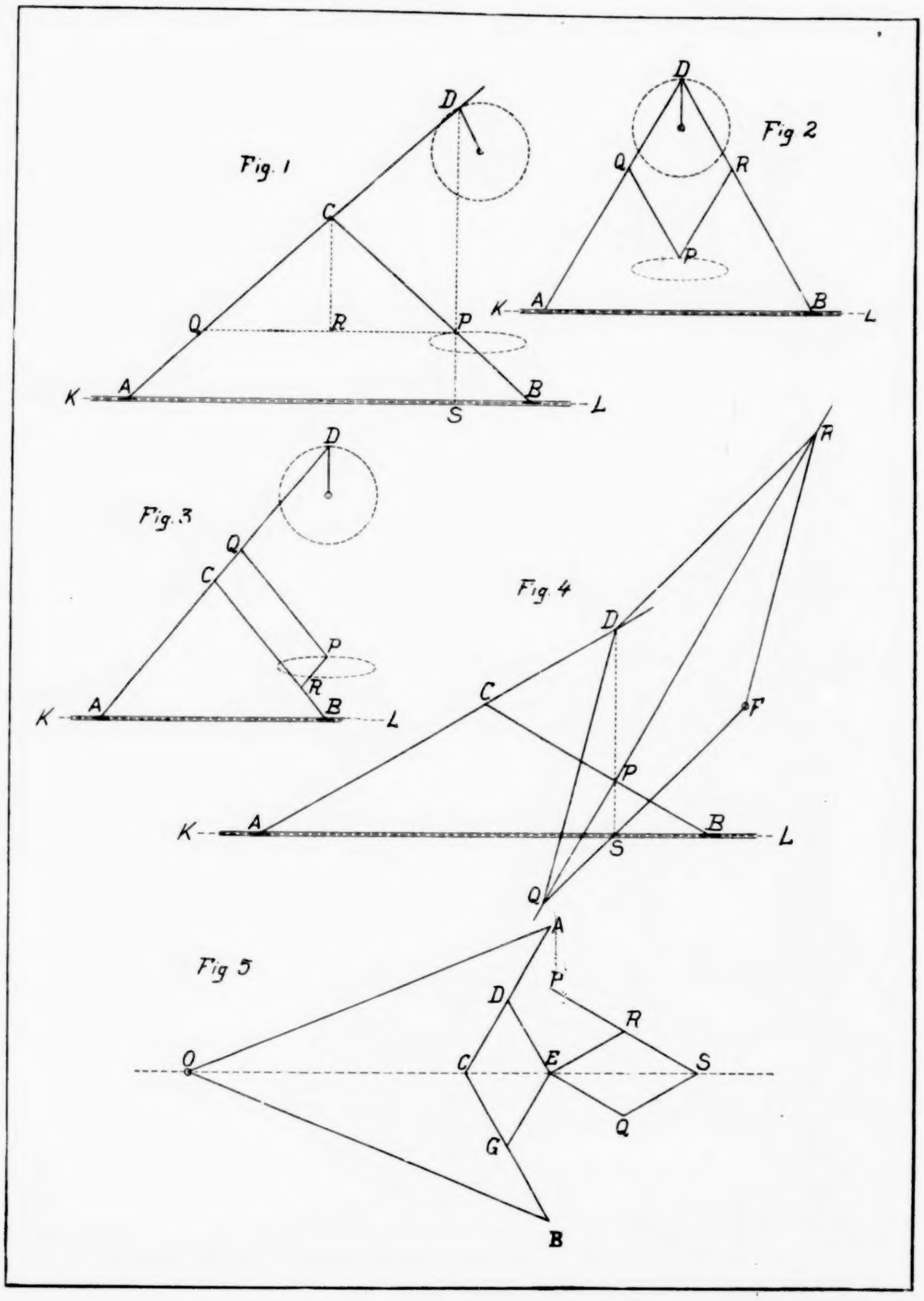
$$= \frac{(x-a)^{n+1}}{n!} \lambda' \theta'^n f^{n+1}\,[x + \theta'(a - x)]\,,$$

in which, as before, all we know of λ' and θ' is that $|\lambda'| < 1$ and $0 < \theta' < 1$, and if x and a are real λ' 1.

We conclude, therefore, that in order to derive the Taylor-Cauchy formula for either real or complex variables, it is only necessary to differentiate the theorem of mean value

$$\frac{fx - fa}{x - a} = \lambda f' [x - \theta (a - x)]$$

n times with respect to a, regarding λ and θ as constants, which however change their values during differentiation, but always remain such that $|\lambda| < 1$, $0 < \theta < 1$, and λ 1 when x and a are real.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E.N.W.

ON LINKAGES FOR TRACING CONIC SECTIONS.

By Mr. Wendell M. Strong, Ithaca, N. Y.

As the linkages described in this paper are believed to be new methods of tracing conic sections, it is thought worth while to offer a brief account of them.

1. *To trace the orthogonal projection of a given curve; and, in particular, of the circle.*

If the intersection of two planes be made the axis of abscissas, the orthogonal projection upon the second plane of any curve lying in the first is given by multiplying the ordinates of this curve by the cosine of the angle between the planes. Since the smaller of the adjacent dihedral angles is usually taken as the angle of the planes, the angle may vary between the limits zero and $\frac{1}{2}\pi$, and its cosine from unity to zero. Therefore a linkage, to project a curve orthogonally, must reduce the ordinates in any given ratio.

In the linkage of Fig. (1), $AC = BC$; P, Q, and D are points taken on BC, AC, and AC extended, such that $CP = CQ = CD$; thus, QP is parallel to AB; and CR, the medial line of triangle QCP, is perpendicular to QP.

Since, in triangles QCR and QDP, $\frac{QC}{QD} = \frac{1}{2} = \frac{QR}{QP}$, angle QPD is a right angle from the similarity of the triangles, and P lies in the ordinate DS. Moreover P divides the ordinate DS in the ratio $\frac{SP}{SD} = \frac{AQ}{AD} = \frac{BC - CP}{BC + CP}$.

Then P is the orthogonal projection of D when the projecting factor is $\cos\theta = \frac{BC - CP}{BC + CP}$. Therefore, if A and B be constrained to move in the right line KL—by sliding or otherwise—and D move on any given curve, a tracing point at P will describe the orthogonal projection of that curve, KL being the line of intersection of the planes, and $\cos^{-1}\frac{BC - CP}{BC + CP}$ the angle between them.

It is evidently possible, by varying CP between the limits zero and BC, to give θ any desired value from zero to $\frac{1}{2}\pi$. Again, by taking P on CB produced, the curve can be traced on the opposite side of the axis KL, but not with the same generality within the limits of finite links.

When the curve to be projected is a circle, the point D may be guided in the circumference by a third link pivoted to the base, and P will then describe an ellipse of which the semi-major axis equals the radius of the circle and the semi-minor axis equals the radius of the circle multiplied by the cosine of θ. Since the orthogonal projection of similar coaxal curves is another series of similar coaxal curves, it is possible to trace a series of similar coaxal ellipses by varying the length of the third link.

Again, if P be constrained to move in any given curve, D will describe the curve of which this is the orthogonal projection. If P move in a circle, D will describe an ellipse of which the semi-minor axis equals the radius of the circle, and equals the semi-major axis multiplied by cosine θ.

This linkage would be of considerable value as a drawing instrument, since it can describe any ellipse or series of concentric ellipses, and can change in any desired ratio the ordinates of a given curve. The mechanical adjustments would be simple, but a discussion of this side of the subject would perhaps be out of place in this paper.

Two other somewhat similar, and probably new but less simple linkages for orthogonal projection are shown in Fig. (2) and Fig. (3). A and B move in the right line KL, and $QPRD$ in Fig. (2) is a rhombus, while $QPRC$ in Fig. (3) is a parallelogram in which $QP = QD$; in Fig. (3) CD also equals BC.

2. *To trace any Conic Section.*

The linkage of Fig. (1) in combination with a rhombus can be used to describe any conic section regarded as the locus of a point the distance of which from a given straight line is in a constant ratio to its distance from a fixed point.

The linkage $ACDB$ of Fig. (4) is the same as the linkage of Fig. (1), and A and B move freely, as before, upon KL; the point P dividing the ordinates in a constant ratio. Let P be further constrained to lie in one diagonal of a rhombus, the extremities of the other being pivoted at D and F. Then if F be a fixed point, P describes a conic. For P being in the diagonal of the rhombus is equally distant from D and F; and therefore

$$\frac{PS}{PF} = \frac{PS}{PD} = \frac{BP}{2CP} = \text{constant}.$$

The conic described is an ellipse, parabola, or hyperbola according as $BP \gtreqless 2CP$, i. e. $BP \gtreqless \frac{2}{3} BC$.

Since $\dfrac{BP}{2CP} = \dfrac{BC - CP}{2CP}$, and CP can assume any value from zero to BC, the ratio $\dfrac{PS}{PF}$ can vary from infinity to zero. Therefore the linkage can draw

any conic. From the fact that P divides the ordinates of D proportionally, D is seen to describe a conic of the same species as P, and can if desired be made the tracing point.

Another possible form which is better in some mechanical respects, but not perfectly general, is given by transferring the angle of the rhombus from D to P, and making D the describing point lying in the diagonal of the rhombus. Then $DP = DF$, and $\frac{DS}{DF} = \frac{DP + PS}{DP} = \frac{2CP + BP}{2CP}$. If P is taken between B and C this gives an ellipse; if at B, a parabola; if on CB extended, an hyperbola. This ratio however can become as small as one-half only when BP becomes negatively infinite. This difficulty could be overcome by completing the rhombus on AB as diagonal and making use of that one of the new links whose extremity is at B, to carry P.

In the use of a rhombus to constrain one point to be equidistant from two other points, it is necessary that the diagonal link slide freely at the angles Q and R, Fig. (4). A method of obtaining the same result without sliding is by combining Mr. Kempe's angle bisector* with a straight line motion. This would require the replacing of a linkage of five links with one of not less than eleven links, and a description of it would require too much space.

3. *Description of the Hyperbola by aid of the Peaucellier Cell.*

Let the cell be as in Fig. (5) in which $OB = OA$, $BC = AC$, $DC = DE = GC = GE = \frac{1}{2}AC$. Then $CDGE$ is a rhombus and E lies in the line AB,

$$\therefore \quad OE^2 = OA^2 - AE^2 = OA^2 - (AC^2 - CE^2),$$

$$OE^2 - CE^2 = OA^2 - AC^2 = \text{a constant.}$$

By making OE the abscissa and CE the ordinate this becomes the equation of a rectangular hyperbola.

Let ER be rigidly attached to DE, and EQ to GE, making angles DER and GEQ each a right angle, and let the rhombus $ERSQ$ be completed, its sides being each equal to DE. Then rhombus $ERSQ$ is rhombus $CDEG$ turned through a right angle, and QR equals CE. If SR be extended to P making RP equal SR, P will lie in the perpendicular to OS erected at E, and PE will equal RQ,

$$\therefore \quad OE^2 - EP^2 = \text{constant.}$$

If O be fixed and S guided in a straight line passing through O, the locus of P is a rectangular hyperbola.

* "How to draw a straight Line." A. B. Kempe, p. 40. Or the same in Nature, Vol. 16. Or "Proceedings of the Royal Society," 1873.

It is not however necessary that the rhombus $ERSQ$ should be of the same size as $CDEG$, and thus it is possible to give $\frac{PE}{CE}$ any desired value. The equation for the curve traced by P then becomes $x^2 - by^2 = c$, which is the general equation of the hyperbola referred to its transverse and conjugate diameters as axes of coordinates.

SOME CONSIDERATIONS ON THE NINE-POINT CONIC AND ITS RECIPROCAL.

By Miss Fanny Gates, Waterloo, Iowa.

In Volume VII of this Journal, Mr. Holgate enunciated the theorem on the nine-point conic in its most general form as follows :—

"Let $ABCD$ be any complete quadrangle whose six sides AB, AC, AD, BC, BD, CD are cut by an arbitrary straight line a, in the points P, Q, R, S, T, V, respectively, and let E, F, H, K, L, M, be the harmonic conjugates of these points with respect to the pairs of vertices of the quadrangle, so that $AEBP$, $AFCQ$, etc., are harmonic ranges. Then a conic may be passed through the six points E, F, H, K, L, M, on which will also lie the three points of intersection X, Y, Z, of the pairs of opposite sides of the quadrangle."

It is well known that the locus of the poles of an arbitrary straight line a with respect to the system of conics through four points, is a conic, (See Smith's Conics, p. 236, Ex. 2), but it does not appear to have been observed that this locus is the nine-point conic determined, as above, by this straight line. This may be readily proved in the following manner :—

Choose AB and CD as the axes of coordinates, and let the equations of AD and BC be

$$ax + by - 1 = 0 \quad \text{and} \quad a'x + b'y - 1 = 0, \quad \text{respectively,}$$

while the equation of the arbitrary straight line a is

$$lx + my - 1 = 0\,.$$

The equation of the system of conics through the four points A, B, C, D, will be

$$(ax + by - 1)(a'x + b'y - 1) - \lambda xy = 0\,.$$

Identify the line a with the polar of an arbitrary point (x', y') with respect to this system of conics, and we obtain as the locus of this point, the conic

$$(2aa' - al - a'l)\,x^2 + (am + a'm - bl - b'l)\,xy - (2bb' - bm - b'm)\,y^2$$
$$- (a + a' - 2l)\,x + (b + b' - 2m)\,y = 0\,.$$

This conic evidently passes through the intersection of AB and CD, and similarly through the intersections of the other pairs of opposite sides of the quadrangle. It cuts the line $x = 0$ a second time at the point where

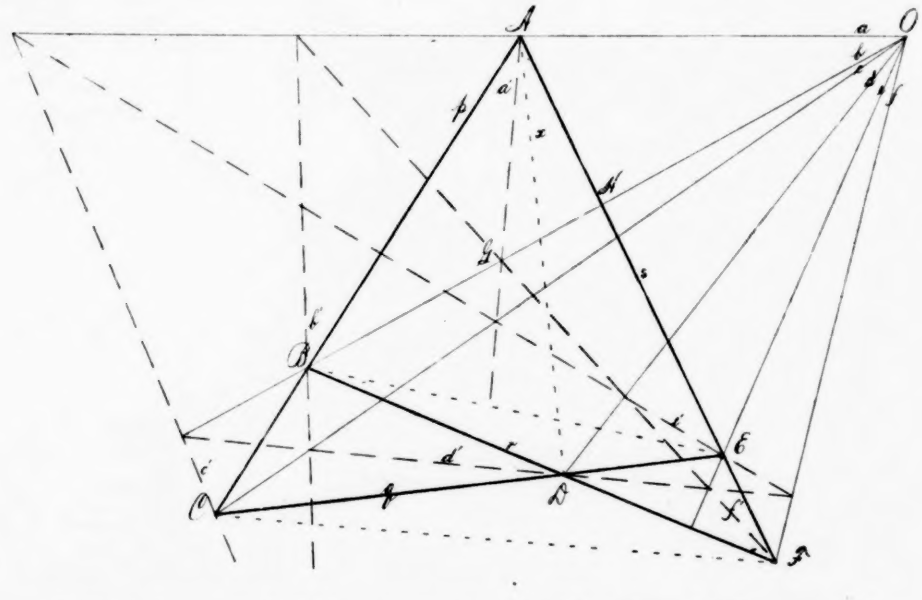
$y = \dfrac{b + b' - 2m}{2bb' - bm - b'm}$. But this point is the harmonic conjugate of the point V (in which the line a cuts the y axis) with respect to the two vertices C and D lying on this side.

In like manner, it may be shown that the locus of the poles of the line a passes through the harmonic conjugate points marked out on the other sides of the quadrangle.

By taking the reciprocal of this general statement of the nine-point conic, we obtain the following interesting theorem which admits a simple geometric proof :—

If the six vertices of a complete quadrilateral be projected from any point O, of the plane, and the harmonic conjugates of these rays with respect to the pairs of sides passing through the same vertex be found, then these six conjugate rays are tangent to one conic, to which also the three diagonals of the quadrilateral are tangent.

Let A, B, C, D, E, F be the six vertices of any complete quadrilateral p, q, r, s, such that A, B, C lie on p, C, D, E on q, B, D, F on r and A, E, F on s.



Project these vertices from any point O of the plane by the rays a, b, c, d, e, f. Let the harmonic conjugates of these rays with respect to the two sides of the quadrilateral passing through the same vertex be a', b', c', d', e', f', respectively.

Then a' and f' will intersect on b. For, if the ray b intersect the side s in H and a' in G, then $OHGB$ is a harmonic range, and if b intersect f' in G' then $OHG'B$ is a harmonic range. Therefore G and G' coincide, that is, a' and f' intersect on b. Likewise b' and f' intersect on a, b' and d' on c, and so also all pairs of similarly situated lines.

Now the rays a', e', c', d', b', f' form a hexagon, such that

a' and e', d' and b' intersect on c,
e' " c', b' " f' " " a,
c' " d', f' " a' " " b;

and since a, b, and c pass through one point O, these lines are tangent to one conic. (Brianchon's Theorem.)

Moreover, if x be any one of the three diagonals of the quadrilateral, say AD, then the rays a', e', f', b', d', x form a hexagon, such that

a' and e', b' and d' intersect on c,
e' " f', d' " x " " d,
f' " b', x " a' " " a;

and since a, c, and d are concurrent, this hexagon circumscribes a conic, that is, the diagonal x, and similarly, each of the other diagonals of the quadrilateral, is tangent to the conic determined by the five rays a', e', f', b', d'. But this is the same conic to which we have shown the ray c' to be tangent. In other words, the six conjugate rays a', b', c', d', e', f' and the three diagonals of the quadrilateral are all tangent to the same conic.

The complete system of tangents to this conic is simply the system of polars of the point O, with respect to the system of conics touching the four sides of the quadrilateral.

This follows directly as the reciprocal of the property proved above, that the locus of the poles of an arbitrary straight line, with respect to the system of conics through four points, is the nine-point conic, determined by the straight line; or it may be readily shown analytically.

The above theorem admits special cases similar to those of the nine-point conic. These arise in accordance with the relative positions of the point O, and the four sides of the quadrilateral.

An analogous configuration in three dimensional space would be the following:—

Let $ABCD$ be any tetrahedron whose edges AB, AC, AD, BC, BD, CD, are projected from an arbitrary point O, not lying in any face of the tetrahedron, by the planes α, β, γ, δ, ε, ζ, respectively, and let the harmonic

conjugates of these planes with respect to the two faces of the tetrahedron passing through the same edge, be α', β', γ', δ', ε', ζ'.

If π be any plane through the point O, it will cut the planes α', β', γ', δ', ε', ζ' in six lines, which are tangent to one conic, to which also are tangent the three lines joining the points where pairs of opposite edges of the tetrahedron are cut by the plane π.

The lines of intersection of the planes α' and ζ', β' and ε', γ' and δ', which pass through the pairs of opposite edges of the tetrahedron, lie in one plane.

The demonstration of this theorem is wholly similar to that enunciated concerning the quadrilateral. In fact, the configuration of the plane theorem is identical with that marked out in the plane of section π.

NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS, *Feb.*, 1894.

BYERLY'S FOURIER'S SERIES AND SPHERICAL, CYLINDRICAL, AND ELLIPSOIDAL HARMONICS.*

The workers in the fields of mathematical science may be roughly classified under three heads. First, there are the pure mathematicians, who cultivate the science with little or no regard to its applications. Many of these devotees, in fact, are willing to pursue lines of investigation which do not obviously lead anywhere; and some few possess or affect a contemptuous disregard for utility in research. They revel in the intricacies of pure analysis and transcendental geometry. Not content with the space and space relations presented to us in nature, they explore all sorts of hypothetical regions and deduce results which would be interesting and possibly useful if humanity were otherwise constituted than it is and existed elsewhere than it does. They work only for the enlargement and refinement of the mathematical mill; whether their attachments will enhance its general efficiency is a question of no importance to them.

Secondly, there is another extreme class, including many physicists, chemists, geologists, etc., who are busied with the investigation of phenomena which must ultimately be reduced to mathematical expression, but which are not yet wholly brought within the domain of competent theories. The devotees of this class are profoundly impressed with the facts they observe. But they distrust mathematical processes and frequently plume themselves on their freedom from all restraints of theory. They cannot listen to the "music of the spheres," especially the music of the molecular spheres of modern physics, in the presence of mathematical machinery. They abominate the details of precise calculations, and are generally content to express the results of their observations by the graphical process, which is not infrequently worked by them to an ingeniously profitless extent. They are quick to discover the salient features and relations of phenomena, but their work usually falls somewhat short of exact generalizations.

Thirdly, between these two extremes there is a smaller class now commonly called mathematical physicists. Their object, like that of the second class, is the interpretation of natural phenomena. But they work always by the aid of mathematical theories, and are content only when they discover and correlate quantitive relations under such theories. Their interest in pure mathematics is restricted to what is obviously useful; and they look with little satisfaction on those branches of the science which have not passed the fact-

* An Elementary Treatise on Fourier's Series and Spherical, Cylindrical, and Ellipsoidal Harmonics, with applications to Problems in Mathematical Physics. By William Elwood Byerly, Ph. D., Boston, U. S. A.: Ginn & Company, Publishers, 1893.

naming and curve-tracing stage. They are the utilitarians and organizers, and to their labors, chiefly, are due the permanent advances in applied mathematics.

Such being the diversity of aims and tastes amongst the workers in mathematical science, it is not often that a new book is of special interest to more than one of the classes named. The subject of this notice, however, is an exception, and must be a source of interest and delight to students in whatever field they may work. Fourier's series, spherical, cylindrical, and ellipsoidal harmonics involve an array of exquisite analysis for the pure mathematician; they afford expression to a wide variety of physical phenomena in ways that cannot fail to be interesting and instructive to the experimentalist; and they indicate the roads along which the mathematical physicist may expect to make further advances.

There is a growing demand for such a book as Prof. Byerly has produced. Hitherto, most students have approached the subject through the original papers of Fourier, Poisson, and Laplace, or through the more recent memoirs of Dirichlet and Riemann. The great work of Heine and the highly condensed chapters in Thomson and Tait's Natural Philosophy are too difficult, and the later works of Todhunter and Ferrers are too special and juiceless for the beginner.

In his plan of presenting the subjects Prof. Byerly has done well, we think, to follow the example of Fourier and Riemann. To the expert this plan may sometimes appear prolix, but is admirably adapted to awaken interest and fix the ideas. It leads directly and naturally from the characteristic differential equations and the conditions of problems to the final integrals. The wonderful properties of these integrals make their discovery one of the most delightful and profitable of studies when the road by which they are reached is not too obscure and circuitous.

The book consists of nine chapters and an appendix. The first of these gives an excellent introduction to the subjects considered, a sort of bird's-eye view, from which the student may get at the outset a fairly good idea of the nature and drift of the enquiry. Chapters II and III are devoted to questions which lead up to Fourier's series and to the proofs of its generality and validity. From a didactic point of view these are the most important chapters in the book; and the author has chosen wisely probably in following pretty closely Dirichlet and Riemann, since the methods of these authorities meet the approval of the majority of mathematicians. We may venture the opinion, however, that Poisson's method of dealing with this delicate branch of analysis is still worthy of study. Indeed, it would seem that what DeMorgan said more than fifty years ago does not now need modification—" Further to verify

the preceding methods," (of development of functions in trigonometric series) " I add one which is of frequent use in the writings of Poisson, and which I consider much the best adapted of any to give a sound view of the subject, as soon as the new and difficult considerations which it introduces have become familiar."*

Chapter IV is devoted to applications, wherein numerous capital illustrations of the use of Fourier's series and integrals are worked out. Chapters V and VI treat of zonal and spherical harmonics ; VII of cylindrical harmonics (Bessel's functions) ; and VIII of curvilinear co-ordinates and ellipsoidal harmonics. The final chapter, by Dr. Maxime Bôcher, gives an interesting though brief history of progress in the development of the subject. The appendix gives useful numerical tables of surface zonal harmonics, hyperbolic functions, roots of Bessel's functions, and values of the latter functions themselves.

A noteworthy feature of the work, a feature which helps much to render it an elementary work in the best sense, is the number and variety of applications carefully explained in full. This, added to the author's direct and clear demonstrations, make the book an uncommonly readable and useful one. He who cannot catch the spirit of the harmonic analysis by the aid of such a book is a hopeless case.

We have a few small faults to record, in the hope that their mention here may be of use to future book-makers. The first of these relates to the matter of notation. The tendency seems to be inevitable in favor of using the symbol ∂ to indicate partial differentiation ; and whatever may be the intrinsic merits of the notation used by the author, it would have been better, we think, for the cause of science, " to fall in with the procession," and write, for example,

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

instead of

$$D_t u = a^2 D_x^2 u \, .$$

Secondly, although many numerical examples and their answers are given in the book, we have not discovered the numerical details of any one of them. This is a defect common to most treatises, except those devoted to astronomy and geodesy, wherein the art of computing is indispensable. The wayfaring student, whose arithmetic is generally the weakest element in his equipment, is always greatly aided in his journey to a new mathematical region by the elaborate details of a few numerical examples.

Lastly, so good and useful a book would have been better and more useful if it had been supplemented by an index. R. S. Woodward.

* DeMorgan, Differential and Integral Calculus, p. 614.

EXERCISES.

373

PROVE that in an ellipse the area of the triangle formed by the tangents at the points whose eccentric angles are α, β, γ, respectively, is

$$A = ab \tan \tfrac{1}{2} (\alpha - \beta) \tan \tfrac{1}{2} (\beta - \gamma) \tan \tfrac{1}{2} (\gamma - \alpha) .$$

[*F. P. Matz.*]

374

SHOW without resorting to the Differential Calculus that the two parabolas whose equations are $y^2 = ax$ and $x^2 = by$ intersect at an angle

$$\theta = \tan^{-1} [3a^{\frac{1}{3}}b^{\frac{1}{3}}/2(a^{\frac{2}{3}} + b^{\frac{2}{3}})] .$$

[*F. P. Matz.*]

375

IF the hodograph of a curve described under constant acceleration be a parabola in which the radius-vectors are drawn from the focus, the intrinsic equation of the curve described is

$$s = c \int \sec^5 \tfrac{1}{2}\varphi \, d\varphi .$$

[*F. P. Matz.*]

376

AT each end of a horizontal base-line $2a$ the angle of elevation of a monument is θ, and at the middle point of this base-line the angle of elevation is φ. Prove that the height of the monument is

$$H = \frac{a \sin \varphi \sin \theta}{\sqrt{[\sin (\varphi + \theta) \sin (\varphi - \theta)]}} .$$

[*F. P. Matz.*]

377

PROVE that the rectangle under the focal distances of the origin in the conic represented by the equation $ax^2 + 2hxy + by^2 = 2y$, is $R = 1/(ab - h^2)$.

[*F. P. Matz.*]

378

A SPHERICAL shrapnel-shell moving at a height $h = 100$ feet above the earth and with a velocity $V = 1700$ feet per second, explodes scattering equably with a velocity $v = 1600$ feet per second its contents and fragments. Draw a curve bounding that part of the earth's surface on which the contents and fragments of the exploded shell have fallen. [*F. P. Matz.*]

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